Influence of the anisotropy parameter Δ on the spectrum of the generalized *q*-symmetrized Harper equation

2003 J. Phys. A: Math. Gen. 36 2077

(http://iopscience.iop.org/0305-4470/36/8/306)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **36** (2003) 2077–2086 PII: S0305-4470(03)37728-5

Influence of the anisotropy parameter Δ on the spectrum of the generalized *q*-symmetrized Harper equation

E Papp

Department of Theoretical Physics, West University of Timisoara, RO 1900 Timisoara, Romania

E-mail: epapp@arad.ro

Received 5 June 2002, in final form 8 November 2002
Published 12 February 2003
Online at stacks.iop.org/JPhysA/36/2077

Abstract

The $\Delta \neq 1$ generalization of the q-symmetrized Harper equation is discussed in terms of wavefunctions expressed by Laurent series. Proceeding by recursion leads to a nontrivial Δ-dependent generalization of the characteristic energy polynomial, with a special emphasis on a continuous dependence on the commensurability parameter. The multiplicity parameter which is responsible for the amount of coprime realizations of the commensurability parameter is also accounted for. The present energies have been derived so as to reproduce particular ones obtained before as limiting cases.

PACS numbers: 72.15.Gd, 71.30.+h, 73.21.−b

1. Introduction

The quantum-mechanical description of Bloch electrons on a two-dimensional (2D) square lattice with nearest-neighbour hopping threaded by a transversal and homogeneous magnetic field B is still a quite fascinating problem exhibiting rich structure and unexpected implementations after nearly half a century [1–9]. The typical equation of this problem is the quasi-periodic second-order discrete Harper equation (HE) [1]

$$\varphi_{n+1} + \varphi_{n-1} + 2\Delta \cos(n\hbar^* + \theta_2)\varphi_n = E\varphi_n \tag{1}$$

where $n \equiv n_1$ is an integer and $\hbar^*$ has the meaning of a cyclic parameter. One has $\hbar^* = 2\pi\beta$, where β is a commensurability parameter expressing the number of magnetic flux quanta per unit cell. So $\beta = Ba^2e/h$, where $-e$ and a denote the electron charge and the lattice spacing, respectively. The related Brillouin phases are denoted by $\theta_l = k_l a$ $(l = 1, 2)$, while the space discretization is performed via $x_l = n_l a$. The anisotropy parameter Δ discriminates between metallic $(\Delta < 1)$ and insulator $(\Delta > 1)$ phases. Equation (1) lies at the confluence of several research fields, such as superconductivity proceeding in terms of linearized Ginzburg–Landau equations [10], the d-wave superconductivity with a magnetic field [11], level statistics in quantum systems with unbounded diffusion [12], critical quantum chaos [13], anomalous diffusion of wave packets in quasi-periodic chains [14], localization length and metal–insulator phase transitions [15] and last but not least the quantum Hall effect [16, 17] and interactions in Aharonov–Bohm cages [18]. Patterns concerning magnetoresistance oscillations of the 2D electron gas [19] and especially the quantized Hall conductance [20] can be traced back to the self-similar nested band energy spectrum of (1), i.e. the celebrated Hofstadter butterfly [4]. The same concerns the 3D version of (1), as explained recently [21]. Such promising issues motivate us to perform further generalizations, now by focussing our attention on the q-symmetrized HE (qSHE) [22–26], which exhibits the symmetry of the quantum group $sl_q(2)$. Accordingly, $q \equiv q(\hbar^*) = \exp(\mathrm{i}\hbar^*/2)$ plays the role of the pertinent deformation parameter. This latter equation works at criticality, i.e. at the $\Delta = 1$ metal–insulator phase transition point and serves as the middle band description of Bloch electrons. However, a systematic study of magnetic properties relying on exact energy solutions of (1) as well as on arbitrary values of the Δ parameter is still missing. We shall then use this opportunity to analyse the $\Delta \neq 1$ generalization

$$\mathrm{i}\left(\frac{1}{z} + \Delta q z\right)\psi(qz) - \mathrm{i}\left(\frac{z}{q} + \frac{\Delta}{z}\right)\psi\left(\frac{z}{q}\right) = E\psi(z) \tag{2}$$

of the qSHE, now in terms of Laurent series (see (4)). This leads to characteristic energy polynomials depending continuously on both $\hbar^*$ and Δ (see (22)–(24)), which represents the main result of this paper.

2. Preliminaries and notations

We would like to say that both (1) and (2) originate from the 2D dispersion law $\varepsilon_d = \varepsilon_1 \cos\theta_1 + \varepsilon_2 \cos\theta_2$, where $\varepsilon_2 = \Delta\varepsilon_1$. The Hamiltonian concerning (2) is then produced by applying the Peierls substitution $k_l \to -\mathrm{i}\partial/\partial x_l + eA_l/\hbar$ to ε_d, which works in terms of the wavefunction

$$\Psi(\vec{x}) = \exp(\mathrm{i}\vec{k}\cdot\vec{x})\widetilde{\varphi}(x_1 + x_2). \tag{3}$$

The vector potential reads $A_l = (-1)^l B(x_1 + x_2 + \alpha_l a)$, which is reminiscent of the 'chiral' gauge (see appendix C in [23] and references therein). Repeating the same steps as before (see, e.g., (A1)–(A10) in [26]) and inserting $\theta_1 = \theta_2 = \pi/2$ and $\alpha_2 = -\alpha_1 = 1/2$ gives (2), this time by preserving the Δ parameter as it stands from the very beginning. We then have to deal with the Laurent series

$$\psi(z) = \sum_{n=-\infty}^{\infty} c_n z^n \tag{4}$$

where $c_n = \widetilde{\varphi}(na) = \widetilde{\varphi}_n$ and $n = n_1 + n_2$. This differs from (1), which relies on the Landau gauge $\vec{A} = (0, Bx_1, 0)$, whereas $\Psi(\vec{x})$ is replaced by $\exp(\mathrm{i}k_2x_2)\varphi(x_1)$. In other words (1) and (2) have to be related by a gauge transformation proceeding in conjunction with an additional inter-connection such that $n_1 \leftrightarrows n$. Similar goals have been discussed, but in a rather involved mathematical manner before [27], by resorting to a generalized formulation of the Bethe-ansatz. Nevertheless, a transparent conversion of (1) into (2), as done for $\Delta = 1$ [28], is still an open problem.

So far the qSHE has been solved by applying a Bethe-ansatz method [22]. In addition, explicit solutions can also be derived by resorting, e.g., to the q-calculus (for more details see, e.g., [29, 30]), which results again in typical three-term recurrence relations [26, 31]. To this aim rational $\beta = P/Q$ values of the commensurability parameter have to be invoked, where P and Q are coprime integers. One would then have $q^{2Q} = 1$, which also indicates that $\widetilde{\varphi}_n$ has to fulfil the periodic boundary condition $\widetilde{\varphi}_n = \widetilde{\varphi}_{n+Q}$. Fixing Q gives an even number, say $2N_s(Q)$, of selected P realizations such as $P_s \equiv P_k^{(Q)}$, which are located symmetrically with respect to the middle point $P = Q/2$. So $P_k^{(Q)} + P_{Q-k}^{(Q)} = Q$, where $k = 1, 2, \ldots, N_s(Q)$. One has, e.g., $N_s(3) = N_s(4) = N_s(6) = 1$, $N_s(5) = N_s(8) = N_s(10) = 2$, but $N_s(7) = N_s(9) = 3$ etc. Of course, inserting $P = P_k^{(Q)}$ yields selected q and $\hbar^*$ realizations having the form

$$q_s \equiv q_k^{(Q)} = \exp\left(\mathrm{i}\pi P_k^{(Q)}\big/Q\right) = -1\big/q_{Q-k}^{(Q)} \tag{5}$$

and $\hbar_s^* \equiv \hbar_k^{*(Q)} = 2\pi P_k^{(Q)}\big/Q$, respectively. After these preliminaries we have to look for a reliable continuous extrapolation for which $\beta \in [0, 1]$, which amounts to considering that $P \in [0, Q]$.

3. The parameter dependence of energy polynomials

Now the problem is to derive a characteristic Q-degree energy polynomial $\widetilde{P}^{(Q)}(E; q, \Delta)$, which should produce the energy eigenvalues of (2) via

$$\widetilde{P}^{(Q)}(E; q, \Delta) = 0 \tag{6}$$

now by looking for a continuous description for which $\beta \in [0, 1]$. We shall denote the roots of this polynomial by $E = E_j^{(Q)}(q, \Delta)$, where $j = 1, 2, \ldots, Q$. It is understood that such polynomials are normalized such that the coefficient of E^Q is unity. Such inherent interpolations producing energies depending continuously on $\hbar^*$ have their own interest. First, they are useful in order to make related symmetry attributes clearer. Second, they serve for the very evaluation of $\hbar^*$ derivatives of energies, which are of interest for an updated study of magnetization effects. Explicit energy polynomials have been discussed in detail recently for $\Delta = 1$ [26]. These latter results are used as inputs for subsequent $\Delta \neq 1$ generalizations. Choosing, e.g., $Q = 5$, we obtain

$$\widetilde{P}^{(5)}(E; q, 1) = E(E^4 - E^2(8 - \Gamma_2 - \Gamma_4 - \Gamma_6 - \Gamma_8) + \Gamma_{12} + \Gamma_{10} - 3\Gamma_8 - \Gamma_6 - 4\Gamma_2 + 12) \tag{7}$$

in which

$$\Gamma_n = \Gamma_n(q) = q^n + 1/q^n = 2\cos(n\hbar^*/2) \tag{8}$$

where n is an integer. We have to realize that Γ_n terms get implemented exclusively with even subscripts, which also means that the $\hbar^*$ dependence of the energy is symmetric with respect to $\hbar^* = \pi$. Furthermore, inserting $q = q_k^{(Q)}$ into $\widetilde{P}^{(Q)}(E; q, \Delta)$ yields $N_s(Q)$ distinct polynomial realizations such as $\widetilde{P}_k^{(Q)}(E; \Delta) = \widetilde{P}^{(Q)}\big(E; q_k^{(Q)}, \Delta\big)$. The interesting point is that these latter polynomials can also be derived identically in terms of the HE by resorting to the method of the secular equation [5, 32] or to the transfer matrix approach [4, 33]. In the latter case explicit polynomials have also been derived recently for $Q = 1 - 8$ [34]. Generalizations of Bethe-ansatz equations mentioned before can also be applied [27], but this method is hardly tractable in practice. Accounting for well-known attributes of the HE, we shall then consider that the pertinent energy bands are generated by the equation

$$\widetilde{P}_k^{(Q)}(E; \Delta) = \Lambda \equiv 2\cos(\theta_1 Q) + 2\Delta^Q \cos(\theta_2 Q) \tag{9}$$

where, of course, $k = 1, 2, \ldots, N_s(Q)$ and $\theta_{lQ} \in [0, 2\pi]$. In particular, one has

$$\widetilde{P}_k^{(5)}(E; \Delta) = \widetilde{P}_\pm^{(5)}(E; \Delta) = E(E^4 - 5E^2(\Delta^2 + 1) + 5(\Delta^4 + 1) + 2.5\Delta^2(3 \pm \sqrt{5})) \tag{10}$$

for $Q = 5$, in which the '$-$' ('$+$') subscript relies on $P_1^{(5)} = 1$ $\left(P_2^{(5)} = 2\right)$. One sees that (7) and (9) meet together if $P = P_k^{(5)}$ and $\Delta = 1$.

4. Deriving the generalized energy polynomial

Next we have to remember that a reliable non-polynomial wavefunction has been proposed before in order to describe, in this way also, the zero-energy solution to the qSHE [35]. Keeping in mind such results, we shall perform the first step towards solving (2) by integrating it along the unit circle centred at $z = 0$ in the complex z-plane. We obtain

$$\mathrm{i}(1-\Delta)c_0 + \frac{\mathrm{i}}{q}c_{-2}(\Delta - q^2) = Ec_{-1} \tag{11}$$

by virtue of (4), which shows that we can proceed further in terms of the $c_{-1} = 0$ choice. We have to realize that alternative choices such as $c_0 = 0$ or $c_{-2} = 0$ are not suitable, as they would produce questionable zero-energy solutions if $\Delta = q^2$ and $\Delta = 1$, respectively. Inserting (4) into (2), we can again find the three-term recurrence relation

$$-\mathrm{i}Ec_n = c_{n+1}\frac{(q^{2n+2}-\Delta)}{q^{n+1}} + c_{n-1}\frac{(\Delta q^{2n}-1)}{q^n} \tag{12}$$

where $c_0 = 1/(1-\Delta)$ and which reproduces (11) via $n = -1$. Recursions of this kind are also well known from the description of (quasi-) exactly solvable models (see, e.g., [36]). We can then obtain

$$c_n = (-\mathrm{i})^n \prod_{j=0}^{n} \frac{q^j}{q^{2j}-\Delta} R^{(n)} \tag{13}$$

for $n \geqslant 0$, where $R^{(n)} \equiv R^{(n)}(E; q, \Delta)$ is a polynomial of degree 'n' in E. Accordingly

$$R^{(n)} = ER^{(n-1)} + R^{(n-2)}(\Delta\Gamma_{2n-2} - 1 - \Delta^2) \tag{14}$$

where now $n \geqslant 1$, such that $R^{(-1)} = 0$ and $R^{(0)} = 1$. The $n \leqslant -2$ counterparts of the above relationships read

$$c_{-n_0} \equiv \widetilde{c}_{n_0} = (-\mathrm{i})^{n_0}(-1)^{n_0-1} \prod_{j=1}^{n_0-1} \frac{q^j}{q^{2j}-\Delta} L^{(n_0-2)} \tag{15}$$

where $n_0 = |n| \geqslant 2$ and where $L^{(n_0-2)}$ is an energy polynomial of degree $n_0 - 2$. This time we have

$$L^{(n_0)} = EL^{(n_0-1)} + L^{(n_0-2)}\left[\Delta\Gamma_{2n_0} - 1 - \Delta^2\right] \tag{16}$$

where $L^{(n_0)} \equiv L^{(n_0)}(E; q, \Delta)$, such that $L^{(-1)} = 0$ and $L^{(0)} = 1$. Some concrete $R^{(n)}$ examples are given in table 1 for $n = 1$–5.

One readily obtains similar results for $L^{(n_0)}$ polynomials, as shown in table 2 for $n_0 =$ 1–3.

After having arrived at this stage we have to look for a suitable eigenvalue condition for (2). For this purpose we shall use a symmetry requirement such as

$$\widetilde{c}_Q = \Omega_0(Q)c_Q = \frac{\exp(\mathrm{i}\pi(Q-1))}{q^Q}c_Q \tag{17}$$

for $n = n_0 = Q$ and $q^{2Q} = 1$, which shows that $c_n = \widetilde{c}_{n_0}$ if $P = P_{Q-1}^{(Q)} = Q - 1$. There are reasons to say that the ansatz (17) can be viewed as a reasonable generalization of the $\Delta = 1$ Bender–Dunne symmetry [36], i.e. of the interpretation of the wavefunction as the generating

Table 1. Explicit $R^{(n)}$ polynomials for $n = 1$–5.

n	$R^{(n)}(E; q, \Delta)$
1	$R^{(1)} = E$
2	$R^{(2)} = R^{(2)} = E^2 - \Delta^2 - 1 + \Delta\Gamma_2$
3	$R^{(3)} = R^{(3)} = E[E^2 - 2(\Delta^2 + 1) + \Delta(\Gamma_2 + \Gamma_4)]$
4	$R^{(4)} = E^4 - E^2[3(\Delta^2 + 1) - \Delta(\Gamma_2 + \Gamma_4 + \Gamma_6)]$ $+ (\Delta^2 + 1)^2 - \Delta(\Delta^2 + 1)(\Gamma_2 + \Gamma_6) + \Delta^2(\Gamma_8 + \Gamma_4)$
5	$R^{(5)} = E[E^4 - E^2(4(\Delta^2 + 1) - \Delta(\Gamma_2 + \Gamma_4 + \Gamma_6 + \Gamma_8))$ $+ 3(\Delta^2 + 1)^2 - \Delta(\Delta^2 + 1)(2\Gamma_2 + \Gamma_4 + \Gamma_6 + 2\Gamma_8)$ $+ \Delta^2(\Gamma_{12} + \Gamma_{10} + \Gamma_8 + \Gamma_6 + 2\Gamma_4)]$

Table 2. Explicit $L^{(n_0)}(E; q, \Delta)$ polynomials for $n_0 = 1$–3.

n_0	$L^{(n_0)}$
1	$L^{(1)} = E$
2	$L^{(2)} = E^2 + \Delta\Gamma_4 - 1 - \Delta^2$
3	$L^{(3)} = E[E^2 - 2(\Delta^2 + 1) + \Delta(\Gamma_4 + \Gamma_6)]$

function of energy polynomials. The behaviour of expansion coefficients for $n \geqslant Q$ and $n \leqslant Q$ can also be easily established. Indeed, there are conversion factors such as $\Omega^{(\pm)}(n)$, such that $\widetilde{c}_n = \Omega^{(\pm)}(n)c(n)$ and $\Omega^{(+)}(Q) = \Omega^{(-)}(Q) = \Omega_0(Q)$, where $n = n_0 \geqslant 2$. The '+' ('$-$') superscript proceeds for $n \geqslant N_0$ ($n \leqslant N_0$), where $N_0 \geqslant 2$. For this purpose we have to apply 'raising' relationships such as

$$\Omega^{(+)}(n_2) = \frac{1}{q^{n_2-N_0}} \frac{\prod_{j=1}^{n_2-N_0}(q^{2N_0+2j} - \Delta)}{\prod_{j=0}^{n_2-N_0-1}(\Delta - q^{2N_0+2j})} \Omega^{(+)}(N_0) \tag{18}$$

proceeding in accord with (11), where $n_2 > N_0 \geqslant 2$. 'Lowering' relationships can be established in a similar manner as

$$\Omega^{(-)}(n_1) = \frac{1}{q^{N_0-n_1}} \frac{\prod_{j=0}^{N_0-n_1-1}(\Delta q^{2N_0-2j} - 1)}{\prod_{j=1}^{N_0-1}(1 - \Delta q^{2N_0-2j})} \Omega^{(-)}(N_0) \tag{19}$$

now for $2 \leqslant n_1 < N_0$. We can now obtain

$$\Omega^{(+)}(n) = \exp(\mathrm{i}\pi(n-1)) \frac{(\Delta - q^{2n})}{q^n(\Delta - 1)} \tag{20}$$

and

$$\Omega^{(-)}(n) = \exp(\mathrm{i}\pi(n-1)) \frac{(\Delta - 1)q^n}{(\Delta q^{2n} - 1)} \tag{21}$$

by virtue of the $N_0 = Q$-fixing, so that $\Omega^{(-)}(n) \equiv 1/\Omega^{(+)}(-n)$.

We shall then proceed further by establishing the polynomial one looks for via $\widetilde{P}^{(Q)}(E; q, \Delta) = 0$, which reflects, of course, the incorporation of the middle band description needed. So we are ready to make the identification

$$\widetilde{P}^{(Q)}(E; q, \Delta) = T(\widetilde{c}_Q - \Omega_0(Q)c_Q) = \frac{(1-\Delta)^2}{\mathrm{i}^Q} \prod_{j=1}^{Q-1} \frac{q^{2j} - \Delta}{q^j} (\widetilde{c}_Q - \Omega_0(Q)c_Q) = 0 \tag{22}$$

Table 3. Explicit $\widetilde{P}^{(Q)}(E; q, \Delta)$-polynomials for $Q = 1$–5.

Q	$\widetilde{P}^{(Q)}$
1	$\widetilde{P}^{(1)} = E$
2	$\widetilde{P}^{(2)} = E^2 - 2(\Delta^2 + 1) + \Delta(2 + \Gamma_2)$
3	$\widetilde{P}^{(3)} = E[E^2 - 3(\Delta^2 + 1) + \Delta(\Gamma_2 + \Gamma_4 + 2)]$
4	$\widetilde{P}^{(4)} = E^4 - E^2[4(\Delta^2 + 1) - \Delta(\Gamma_2 + \Gamma_4 + \Gamma_6 + 2)]$
	$+ \Delta^2(\Gamma_8 + 3\Gamma_4) + (\Delta^2 + 1)[2(\Delta^2 + 1) - \Delta(\Gamma_2 + \Gamma_4 + \Gamma_6 + 2)]$
5	$\widetilde{P}^{(5)} = E[E^4 - E^2(5(\Delta^2 + 1) - \Delta(\Gamma_2 + \Gamma_4 + \Gamma_6 + \Gamma_8 + 2))$
	$- 2\Delta(\Delta^2 + 1)(\Gamma_2 + \Gamma_4 + \Gamma_6 + \Gamma_8 + 2) + 5(\Delta^4 + 1)$
	$+ \Delta^2(\Gamma_{12} + \Gamma_{10} + \Gamma_8 + 3\Gamma_6 + 4\Gamma_4 + 10)]$

in accord with (17), where the T-factor serves solely to normalize the coefficient of E^Q in the energy polynomial to unity. The generalized polynomial is then given by

$$\widetilde{P}^{(Q)}(E; q, \Delta) = R^{(Q)}(E; q, \Delta) - (1 - \Delta)^2 L^{(Q-2)}(E; q, \Delta) \tag{23}$$

which reproduces concrete results concerning both $\widetilde{P}^{(Q)}(E; q, 1)$ [26]— and $\widetilde{P}_k^{(Q)}(E, \Delta)$—[34] limits. So we are led to explicit q-dependent energy polynomials as presented in table 3 for $Q = 1$–5.

Other cases can be treated in a similar manner. So we succeeded in establishing $\widetilde{P}^{(Q)}(E; q, \Delta)$ polynomials producing discrete $\widetilde{P}_k^{(Q)}(E, \Delta)$ realizations via $q = q_k^{(Q)}$. Such findings serve to the derivation of extrapolated energies depending continuously on $\hbar^* \in [0, 2\pi]$ via $P \in [0, Q]$.

5. Further properties

The $E = E_j^{(Q)}(q, \Delta) \equiv E_j^{(Q)}(\hbar^*; \Delta)$ roots of above polynomials can be ordered as

$$E_1^{(Q)} < E_2^{(Q)} < \cdots < E_{Q-1}^{(Q)} < E_Q^{(Q)} \tag{24}$$

such that $E_j^{(Q)}(q, \Delta) = -E_{Q-j+1}^{(Q)}(q, \Delta)$ by virtue of the energy-reflection symmetry [37]. Plotting these energies versus $\hbar^* \in [0, 2\pi]$, we can realize that $E_j^{(Q)}(\hbar^*; \Delta) = E_j^{(Q)}(2\pi - \hbar^*; \Delta)$, where $q(2\pi - \hbar^*) = -1/q(\hbar^*)$. We have to remark that energies implied by above polynomials are smooth varying functions of $\hbar^*$ if $\Delta \neq 1$, as shown, e.g., by the solid ($\Delta = 2$) energy curves displayed in figure 1, for $Q = 4$. In contrast, the dotted curves in the same figure show that the $\Delta = 1$ description is characterized by symmetrically located contact points between adjacent levels [26]. Such points are visualized by jumps between lateral $\hbar^*$ derivatives of energies, as shown by the dotted curves in figure 2. Thus we can observe local cusplike maxima (minima) for energy curves located below (above) the $E = 0$ axis. Except the $E = 0$ levels implied by odd Q values if $j = (Q + 1)/2$, we have $E_j^{(Q)} \sim |\Delta - 1|$ for $\hbar^* = 0$ and $\hbar^* = 2\pi$ as well. This behaviour is illustrated in table 4 for $Q = 2$–5.

There is $E_j^{(Q)}(1, \Delta) = E_j^{(Q)}(-1, \Delta)$, where the $\pm$ signs quoted in table 4 are well understood in terms of (24). We see that the $\hbar^* = 0$ limits displayed above are expressed by irrational numbers, the complexity of which increases with Q. Such results are able to serve as signatures of hierarchical attributes of the energy-spectrum, this time for $\Delta \neq 1$.

It is obvious that by now $\partial E_j^{(Q)} \big/ \partial \hbar^*$ derivatives can be easily established, as displayed by the solid curves in figure 2, for $Q = 4$. In particular, e.g., $\partial E_j^{(4)} \big/ \partial \hbar^* = \varepsilon_j / \sqrt{3}$ for $\hbar^* = \pi/2$ and $\Delta = 1$, but

$$\partial E_j^{(4)} \big/ \partial \hbar^* |_{\hbar^* = \pi/2} = \varepsilon_j [\gamma_1(\delta_{j,1} + \delta_{j,4}) + \gamma_2(\delta_{j,2} + \delta_{j,3})] \tag{25}$$

if $\Delta = 2$, where $\gamma_1 \cong 0.758\,923$, $\gamma_2 \cong 0.561\,516$ and $\varepsilon_j = 1\ (-1)$ for $j = 1, 2\ (3, 4)$.

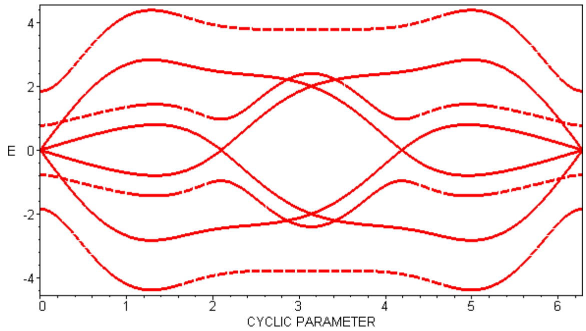

Figure 1. The $\hbar^*$ dependence of the four $Q = 4$ energy levels for $\Delta = 1$ (the solid curves approaching the $E = 0$ axis) and $\Delta = 2$ (dotted curves).

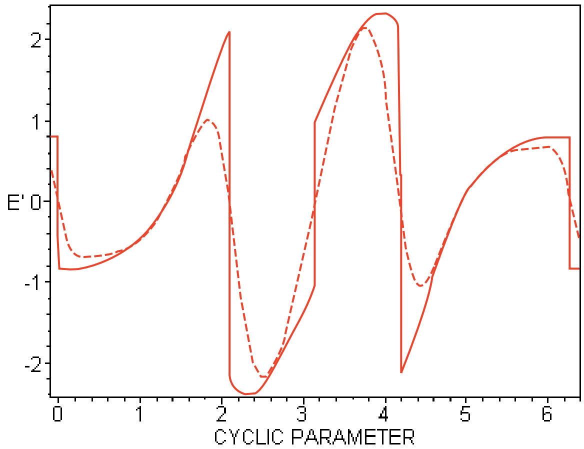

Figure 2. The $\hbar^*$ dependence of the derivatives $E' = \partial E_2^{(4)}(\hbar^*; 1)/\partial\hbar^*$ (solid curve) and $E' = \partial E_2^{(4)}(\hbar^*; 2)/\partial\hbar^*$ (dotted curve). We find that jumps located at $\hbar^* = 0, 2\pi/3, 4\pi/3$ and 2π if $\Delta = 1$, but the derivative is continuous for $\Delta = 2$.

Table 4. The $\hbar^* = 0$ limit of $E_j^{(Q)}(q, \Delta)$ energies for $Q = 2$–5.

Q	$E_j^{(Q)}$
2	$E_j^{(2)}(1, \Delta) = \pm\sqrt{2}\|\Delta - 1\|$
3	$E_j^{(3)}(1, \Delta) = \pm\sqrt{3}\|\Delta - 1\|$
4	$E_j^{(4)}(1, \Delta) = \pm\sqrt{2 \pm \sqrt{2}}\|\Delta - 1\|$
5	$E_j^{(5)}(1, \Delta) = \pm\sqrt{5 \pm \sqrt{20}}\|\Delta - 1\|$

The Δ dependence of present energies is also of interest. Choosing once again $Q = 4$ and inserting, e.g., $\hbar^* = 0.005$, we find that energies displayed versus Δ look like straight lines exhibiting the tendency to meet together at $\Delta = 1$ and $E = 0$, as indicated by the straight lines in figure 3. More precisely, there is $\partial E_3^{(4)}\big/\partial\Delta \cong 0.764\,54\,\mathrm{sgn}(\Delta - 1)$ and $\partial E_4^{(4)}\big/\partial\Delta \cong 1.845\,74\,\mathrm{sgn}(\Delta - 1)$, where $\hbar^* = 0.005$. However, the dotted curves in figure 3 show that this behaviour is increasingly lost for larger $\hbar^*$ values, now for $\hbar^* = 0.2$.

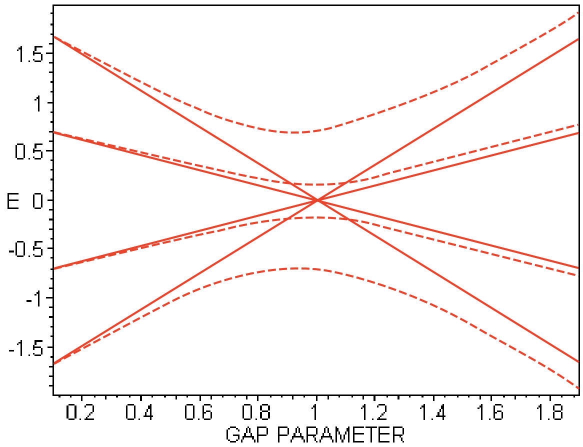

Figure 3. The Δ dependence of the four $Q = 4$ energy levels for $\hbar^* = 0.005$ (solid lines) and $\hbar^* = 0.2$ (dotted curves).

Without considering further details, we have to realize that the wavefunction can be reduced to a rational function by virtue of the periodicity $c_{n\pm 2Q} = c_n$ of expansion coefficients. Accordingly, one has $\Omega^{(\pm)}(n \pm 2Q) = \Omega^{(\pm)}(n)$. The whole interval $n \in I \equiv [2, \infty)$ can then be expressed as a union over periodicity intervals such as $I = \bigcup_{i=0}^{\infty} I_i$, where $I_i = [2 + 2iQ, (2i + 2)Q + 1]$ and $i = 0, 1, 2, \ldots$. Then monomials such as z^n and $1/z^n$ get multiplied by a series such as $S = 1 + z^{2Q} + z^{4Q} + \cdots$, which can be summed up to $S = 1/(1 - z^{2Q})$ if $|z| < \exp(1/2Q)$. Under such circumstances the wavefunction (4) can be expressed as

$$\psi(z) = \psi^{(Q)}(q, z) = \frac{1}{1-\Delta}\left(1 - \frac{\mathrm{i}qEz}{q^2 - \Delta}\right) + \frac{1}{1 - z^{2Q}} \sum_{n=2}^{2Q+1} c_n \left(z^n + \frac{1}{z^n}(\Omega^{(+)}(n)\theta(n - Q) + \Omega^{(-)}(n)\theta(Q - n))\right) \tag{26}$$

in which $\theta(0) = 1/2$. It is clear that

$$\psi(z) \to \frac{1}{1-\Delta} + z(c_1 - c_{2Q+1}) = \frac{1}{1-\Delta} \tag{27}$$

if $z \to \infty$. This indicates that the normalization interval should be finite, unless the wavefunction is renormalized. Realizations such as $\psi^{(Q)}_{k,j}(z)$ can also be obtained immediately by inserting $q = q^{(Q)}_k$ and $E = E^{(Q)}_j(q, \Delta)$.

A further point concerns the Aubry duality [32, 38], which has been discussed in some detail in terms of (1). Now we are in a position to derive a reliable manifestation of this duality by using, at least for the moment, the Fourier transform

$$\psi(z) = \int_{-\infty}^{\infty} \Phi(s) \exp(\mathrm{i}sz)\, \mathrm{d}s. \tag{28}$$

Indeed, inserting (28) into (2) yields the second-order differential-difference equation

$$E\frac{\mathrm{d}}{\mathrm{d}s}\Phi(s) = \frac{1}{q}\Phi\left(\frac{s}{q}\right) - \Delta\frac{\mathrm{d}^2}{\mathrm{d}s^2}\Phi\left(\frac{s}{q}\right) - q\Delta\Phi(qs) + \frac{\mathrm{d}^2}{\mathrm{d}s^2}\Phi(qs) \tag{29}$$

under the assumption that surface terms are zero. We realize immediately that the $E = E(q, \Delta)$ solution can be rewritten equivalently as $-\Delta E(1/q, 1/\Delta)$, where, of course,

$E(q, \Delta) = E(1/q, \Delta)$. Thus we found a typical manifestation of the Aubry duality, with the understanding that more intricate convergence properties are left aside.

6. Conclusions

In conclusion, we succeeded in establishing reliable $\Delta \neq 1$ generalizations of characteristic energy polynomials concerning the Harper equation towards continuous values of the commensurability parameter. This opens the way to establish $\hbar^*$ derivatives of energies, which have their own intrinsic interest. For this purpose (2) has been solved in terms of Laurent series by using a generalized Bender–Dunne ansatz as given by (17). Moreover, a rational reduction of such series can also be done, as shown by (26). The multiplicity parameter $k = 1, 2, \ldots, N_s(Q)$, which seems to have been ignored before, has also been accounted for. The energy polynomials referred to above produce discrete realizations via $\hbar^* \equiv \hbar_k^{*(Q)}$. So we are now in a position to perform updated studies of magnetic properties with regard to both metallic and insulator phases. For this purpose fixed Q-values can be invoked, in which case the density of states can be expressed again in terms of complete elliptic integrals of the first kind [5, 34, 39]. We emphasize that present results are favoured on general theoretical grounds, but we have to be aware that other versions concerning continuous extrapolations of energy polynomials could also be established by resorting to transfer matrices or to the method of the secular equation. Such alternative results identically reproduce the present ones in selected $q = q_k^{(Q)}$ points, as one might expect. Then the insensitivity of thermodynamic properties with respect to such versions remains to be clarified in several respects. A similar behaviour concerns the Hall conductance, which is (in contrast with the edge states) the same both in commensurate and incommensurate cases [40]. We can then say that present extrapolations towards continuous values of the commensurability parameter can also be viewed as expressing inherent incorporations of incommensurability effects, i.e. of irrational values of the P parameter.

Acknowledgments

I would like to thank H D Doebner, F Gebhardt, I Cotaescu and W Bestgen for useful discussions, as well as to CNCSIS/Bucharest for financial support.

References

[1] Harper P G 1955 *Proc. Phys. Soc.* A **68** 874
[2] Rauh A 1975 *Phys. Status Solidi* b **69** K9
[3] Azbel M Ya 1964 *Sov. Phys.–JETP* **19** 634
[4] Hofstadter D R 1976 *Phys. Rev.* B **14** 2239
[5] Wannier G H, Obermair G M and Ray R 1979 *Phys. Status Solidi* b **93** 337
[6] Wilkinson M 1986 *Proc. R. Soc.* A **403** 135
[7] Sokoloff J B 1985 *Phys. Rep.* **126** 189
[8] Rammal R and Bellissard J 1990 *J. Phys. (Paris)* **51** 1803, 2153
[9] Hiramoto H and Kohmoto B 1992 *Int. J. Mod. Phys.* B **6** 281
[10] Wang Y Y, Pannetier B and Rammal R 1987 *J. Physique* **48** 2067
[11] Morita Y and Hatsugai Y 2001 *Phys. Rev. Lett.* **86** 151
[12] Geisel T, Ketzmerick R and Petschel G 1991 *Phys. Rev. Lett.* **66** 1671
[13] Evangelou S N and Pichard J L 2000 *Phys. Rev. Lett.* **84** 1643
[14] Piéchon F 1996 *Phys. Rev. Lett.* **76** 4372
[15] André G and Aubry S 1980 *Ann. Isr. Phys. Soc.* **3** 133
[16] Thouless D J, Kohmoto B, Nightingale M P and den Nijs M 1982 *Phys. Rev. Lett.* **49** 405

[17] Springsguth D, Ketzmerick R and Geisel T 1997 *Phys. Rev.* B **56** 2036
[18] Vidal J, Butaud P, Doucot B and Mosseri R 2001 *Phys. Rev.* B **64** 155306
[19] Gerhards R R and Pfannkuche D 1992 *Surf. Sci.* **263** 324
[20] Albrecht C, Smet J H, von Klitzing K, Weiss D, Umansky V and Schweizer H 2001 *Phys. Rev. Lett.* **86** 147
[21] Koshino M, Aoki H, Kuroki K, Kagoshima S and Osada T 2001 *Phys. Rev. Lett.* **86** 1062
[22] Wiegmann P B and Zabrodin V A 1994 *Phys. Rev. Lett.* **72** 1890
[23] Wiegmann P B and Zabrodin V A 1994 *Nucl. Phys.* B **422** 495
[24] Hatsugai Y, Kohmoto M and Wu Y S 1994 *Phys. Rev. Lett.* **73** 1134
[25] Abanov A G, Talstra J C and Wiegmann P B 1998 *Phys. Rev. Lett.* **81** 2112
[26] Papp E and Micu C 2002 *Phys. Rev.* E **65** 046234
[27] Fadeev L D and Kashaev R M 1995 *Commun. Math. Phys.* **169** 181
[28] Krasovsky I V 1999 *Phys. Rev.* B **59** 322
[29] Gasper G and Rahman M 1990 *Basic Hypergeometric Series* (Cambridge: Cambridge University Press)
[30] Papp E 1997 *The Derivation of q Analogs for the Radial Schrödinger Equation in N Space Dimensions* (New York: Nova Science)
[31] Micu C and Papp E 1997 *Mod. Phys. Lett.* B **11** 773
[32] Hatsugai Y and Kohmoto M 1990 *Phys. Rev.* B **42** 8282
[33] Kohmoto M, Kadanoff L P and Tang C 1983 *Phys. Rev. Lett.* **50** 1870
[34] Papp E, Micu C and Szakacs Zs 2002 *Int. J. Mod. Phys.* B **16** 3481
[35] Micu C and Papp E 1998 *J. Phys. A: Math. Gen.* **31** 2881
[36] Bender C M and Dunne G V 1996 *J. Math. Phys.* **37** 6
[37] Shifman M and Turbiner A 1999 *Phys. Rev.* A **59** 1791
[38] Kohmoto M 1989 *Phys. Rev.* B **39** 11943
[39] Hasegawa Y, Hatsugai Y, Kohmoto Y M and Montambaux G 1990 *Phys. Rev.* B **41** 9174
[40] Hatsugai Y 1993 *Phys. Rev.* B **48** 11851